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On the Equitable Distribution of Surplus. By SHEPPARD HOMANS, Esq., Actuary of the Mutual Life Insurance Company of New York.

HAVING read with much pleasure and profit the several articles upon this subject, chiefly in the *Assurance Magazine*, from such able contributors as Messrs. De Morgan, Jellicoe, Brown, Sprague, Pattison and others, whose names are known and honoured on this side the Atlantic as well as at home, I have thought that, in return, a brief account of the method adopted in the recent distribution of the surplus of our largest American Life Company might be acceptable to the members of the Institute of Actuaries.

The experience of the Mutual Life Insurance Company of New York has, especially during the past five years, been far more favourable, both in regard to mortality and interest, than was assumed when the premiums were determined on, or than it will be at all prudent to anticipate in future. Assuming a rate of mortality somewhat similar in the aggregate to that represented by the Carlisle table, and 4 per cent. interest, we have experienced during that time not more than 70 per cent. of that mortality and a *net* interest of 6½ per cent. per annum, very nearly. The dividends are declared once in five years, or oftener, and the surplus is divided in accordance with a provision of the charter, which prescribes that the officers "shall credit each member with an *equitable share* of the profits."

A distribution of the large amount of surplus which must necessarily have arisen from conditions so favourable, and yet so novel, in proportion either to premiums paid or to sums assured, or, indeed, by any *usual* method, would not only have been far from equitable, but in consequence a violation of the charter of the Company. Indeed, it soon became manifest that no method hitherto suggested would give results entirely satisfactory, and that an independent investigation was absolutely necessary.

Let us assume in the first place the general correctness of the principle, seemingly recognised by all actuaries, that each participant should be benefited in proportion to the excess of his payments over and above the actual cost of insurance. We shall find, however, that this principle is susceptible of some modification when a portion of the total surplus was due to sources other than the over-payments of present members. The first point, however, after the amount of the general surplus shall have been correctly ascertained, will be to determine what the *actual cost of insurance,* and consequently the over-payment, has been in any and every case. Strictly speaking, this cannot be done until every policy shall have determined by death or otherwise; but, in that case, there could be no distribution of surplus during the "lifetime" of any policy. Practically, however, we can make such a *reserve* for all liabilities as to determine what sum may *safely* be distributed as surplus. The problem now before us, however, is not the *amount* but the *equitable proportions* of the surplus, which we assume really exists. As a general rule the accumulated payments in any policy are, or should be, absorbed—1st, by the actual cost of insurance (whatever that may have been); 2nd, by the *reserve* for the present value, or reinsurance; and, 3rd, by the remainder, which is the over-payment or contribution to the surplus fund. The cost of insurance during any interval of time can only be determined by a careful analysis of the results of experience. The *reserve,* or present value, being placed at the *credit* of the policy-holder, will, *even if too large,* only increase by such excess his equitable share of surplus at the *next* distribution.

The following is a demonstration of the method which I purpose for determining the contribution, or over-payment, of any policy, and which has been adopted by the "Mutual Life" in the recent distribution of surplus:—

Let $(1+i)^t$ = amount of 1 dollar at the end of t years, according to the rate of interest actually realised from the investments of the Company.

$1+a^{t-1}$ = amount of 1 dollar per annum (certain) in advance, for t years, also at the actual rate of interest.

p'_x = premium to assure 1 dollar *for one year*, at the age of x, according to the rates of mortality and interest actually experienced; in other words, the actual cost of insurance *for that time*.

R_{x+n} = amount *actually reserved* at the *last* distribution as the then present value of a policy of 1 dollar issued n years previous, at the age of x.

R'_{x+n+t} = amount actually reserved at the present distribution as the present value of the same policy (*accented* to denote the effect of any change in the scale of *reserves* or present values).*

$\mathfrak{P}_x$ = *effective* annual premium, or the premium actually paid, diminished by a *constant* for expenses and a percentage for commissions. [In the Mutual Life $\mathfrak{P}_x = \frac{95}{100} P_x - \cdot 002S$.] If (as has been the case with the Mutual Life) the *extraneous* profits, arising from lapses, short terms, &c., are sufficient to cover the general expenses, $\mathfrak{P}_x$ (for the *past*) should equal P_x, the premium actually paid.

c_{x+n} = contribution to surplus, or over-payment, by a policy of 1 dollar, issued n years before, at the age of x, for the next t years.

Then, employing the usual notation, the contribution to the surplus of a Company at the end of one year, by l_x persons insured n years before, at the age of x, for 1 dollar each, will be

$$l_{x+n}R_{x+n}(1+i)+l_{x+n}\mathfrak{P}_x(1+i)-d_{x+n}-l_{x+n+1}R'_{x+n+1}=c'_{x+n}l_{x+n}.$$

Substituting for l_{x+n+1}, in this equation, its equivalent $l_{x+n} - d_{x+n}$, we have

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* The present values, or *reserves*, of policies at the last dividend (in 1858) were determined, by valuing the *net* premiums according to the "Gill" Table of Mortality, at 4 per cent. interest, with a margin of 15 per cent.; thus:—

$$R_{x+n} = 1\cdot 15\,[A_{x+n} - p_x(1+a_{x+n})].$$

The *reserves* at the present distribution were based upon a new table of mortality, constructed by the present actuary, at 4 per cent. interest, but by a valuation of the *effective* annual premium ($\mathfrak{P}_x = \frac{95}{100} P_x - \cdot 002S$), thus:—

$$R'_{x+n+5} = 1\cdot 20A'_{x+n+5} - \mathfrak{P}_x(1+a'_{x+n+5}).$$

$$l_{x+n}\mathrm{R}_{x+n}(1+i)+l_{x+n}\mathfrak{P}_x(1+i)-d_{x+n}-(l_{x+n}-d_{x+n})\mathrm{R}'_{x+n+1}=l_{x+n}c'_{x+n}$$
$$=l_{x+n}\mathrm{R}_{x+n}(1+i)+l_{x+n}\mathfrak{P}_x(1+i)-d_{x+n}(1-\mathrm{R}'_{x+n+1})-l_{x+n}\mathrm{R}'_{x+n+1}$$
$$=l_{x+n}c'_{x+n}.$$

Dividing this equation by l_{x+n}, we have, since $\frac{d_{x+n}}{l_{x+n}}=p'_{x+n}(1+i)$, as the contribution *by each person* at the end of the first year,

$$\mathrm{R}_{x+n}(1+i)+\mathfrak{P}_x(1+i)-p'_{x+n}(1+i)(1-\mathrm{R}'_{x+n+1})-\mathrm{R}'_{x+n+1}=c'_{x+n}.$$

At the end of the *second* year the contribution from each person whose policy remained in force at the beginning of that year, will be

$$[\mathrm{R}_{x+n}\ (1+i)+\mathfrak{P}_x(1+i)-p'_{x+n+0}(1+i)\ (1-\mathrm{R}'_{x+n+1})-\mathrm{R}'_{x+n+1}](1+i)+$$
$$\mathrm{R}'_{x+n+1}(1+i)+\mathfrak{P}_x(1+i)-p'_{x+n+1}(1+i)\ (1-\mathrm{R}'_{x+n+2})-\mathrm{R}'_{x+n+2}=c^2_{x+n}=$$
$$\mathrm{R}_{x+n}\ (1+i)^2+\mathfrak{P}_x(1+i)^2-p'_{x+n+0}(1+i)^2(1-\mathrm{R}'_{x+n+1})$$
$$+\mathfrak{P}_x(1+i)-p'_{x+n+1}(1+i)\ (1-\mathrm{R}'_{x+n+2})-\mathrm{R}'_{x+n+2}=c^2_{x+n}.$$

At the end of *five* years (which we will now suppose to be the *bonus period*) the contribution from each person whose policy remained in force at the beginning of the fifth year, will be

$$\begin{aligned}\mathrm{R}_{x+n}(1+i)^5+\mathfrak{P}_x(1+a^4)&-p'_{x+n+0}(1+i)^5(1-\mathrm{R}'_{x+n+1})\\&-p'_{x+n+1}(1+i)^4(1-\mathrm{R}'_{x+n+2})\\&-p'_{x+n+2}(1+i)^3(1-\mathrm{R}'_{x+n+3})\\&-p'_{x+n+3}(1+i)^2(1-\mathrm{R}'_{x+n+4})\\&-p'_{x+n+4}(1+i)(1-\mathrm{R}'_{x+n+5})-\mathrm{R}'_{x+n+5}=c^5_{x+x}.\end{aligned}$$

From the foregoing equations it appears that the contributions, or over-payments of policies during a bonus period, may in general be found thus:—*Credit* each policyholder, 1st, with the amount actually reserved at the last preceding distribution of surplus as the then present value, or reinsurance of the policy; and, 2nd, with the *effective* (or *full*) premiums paid since that time, both sums being accumulated at the actual current rate of interest to the date of the present distribution; and *charge* him, 1st, with the actual cost of the risk to which the Company has been exposed *during the interval*, determined by means of a table representing the rates of mortality and interest actually experienced; and, 2nd, with the amount now reserved as the present value of the policy. The difference between the sum of his *credits* and the sum of his *debits* determines the *over-payment* or *contribution* from the policy proper.

If the amount of surplus at the previous distribution was determined on the assumption that the rates of mortality and interest would be less favourable than they have been *since that time,* a certain portion of the present surplus arising in consequence, clearly belongs to those persons exclusively who were insured previous to the date of that distribution, and who would be defrauded to a certain extent if more recent members were allowed to participate in that portion of the surplus, to which they could have no just claim. Thus the contribution from a former dividend applied to the purchase of a reversionary addition of 1 dollar (or from a paid up policy)—the present value of which at the last distribution was A_{x+n} is, at the end of five years,

$$\begin{aligned}A_{x+n}(1+i)^{5}&-p'_{x+n+0}(1+i)^5(1-A'_{x+n+1})\\&-p'_{x+n+1}(1+i)^4(1-A'_{x+n+2})\\&-p'_{x+n+2}(1+i)^3(1-A'_{x+n+3})\\&-p'_{x+n+3}(1+i)^2(1-A'_{x+n+4})\\&-p'_{x+x+4}(1+i)\,(1-A'_{x+n+5})-A'_{x+n+5}.\end{aligned}$$

By a similar course of reasoning it will be found that the contribution from a former dividend applied to the purchase of an annuity of 1 dollar, acting as an annual reduction of premium, the present value of which, at the last distribution, was $1+a_{x+n}=\Delta_{x+n}$, is, at the end of five years,

$$\begin{aligned}\Delta_{x+n}(1+i)^5-&(1+i)^5(1-p'_{x+n+0}\Delta'_{x+n+1})\\&(1+i)^4(1-p'_{x+n+1}\Delta'_{x+n+2})\\&(1+i)^3(1-p'_{x+n+2}\Delta'_{x+n+3})\\&(1+i)^2(1-p'_{x+n+3}\Delta'_{x+n+4})\\&(1+i)\,(1-p'_{x+n+4}\Delta'_{x+n+5})-\Delta'_{x+n+5}.\end{aligned}$$

Contributions from death policies.

The contributions from a person dying between two distributions—say, for instance, during the first year—is

$$[R_{x+n}(1+i)+\mathfrak{P}_x(1+i)-p'_{x+n+0}(1+i)(1-R'_{x+n+1})-R'_{x+n+1}](1+i)^4$$
$$=c'_{x+n}(1+i)^4$$
$$=R_{x+n}(1+i)^5+\mathfrak{P}_x(1+i)^5-p'_{x+n+0}(1+i)^5(1-R'_{x+n+1})-R'_{x+n+1}(1+i)^4$$
$$=c'_{x+n}(1+i)^4.$$

If death should occur during the *third* year the contribution would be

$$\begin{aligned}R_{x+n}(1+i)^5+&\mathfrak{P}_x(1+i)^5-p'_{x+n+0}(1+i)^5(1-R'_{x+n+1})\\&\mathfrak{P}_x(1+i)^4-p'_{x+n+1}(1+i)^4(1-R'_{x+n+2})\\&\mathfrak{P}_x(1+i)^3-p'_{x+n+2}(1+i)^3(1-R'_{x+n+3})-R'_{x+n+3}(1+i)^2\\&\qquad=c^3_{x+n}(1+i)^2.\end{aligned}$$

If death should occur during the *fifth* year the contribution would be $= c^5_{x+n}$.

The contributions corresponding to each age, and to every year after issue of the policy, were carefully computed by means of the foregoing equations and then tabulated. The proper amounts due on the 1st day of February, 1863 (the date of the present distribution), to each policy then in force (including those lapsed by death since 1858) were then ascertained. The result showed in each case the contribution or over-payment.

The sum total of the contributions thus found showed what portion of the general surplus resulted from the over-payments of existing members (including death policies), and also what other portions resulted from extraneous sources, such as lapses, &c. Nothing could be more just than that the portion first named should be placed at the credit of the original contributors in the proportions paid by each. This was done. The distribution of the remainder of the surplus, however, is not governed by the same principles of equity. It was a fund belonging to the members as a body, and might equitably have been divided amongst them all, *pro rata,* in equal proportions, or amongst them in proportion to premiums paid, or in proportion to sums insured, or, indeed, by *any* method which may have been prescribed by the charter or promised by the Company. Its distribution involves no principle of *mathematical* equity, and hence it should not *necessarily* be divided amongst policy-holders in proportion to their contributions.

In the "Mutual Life" this portion of the general surplus was applied—1st, as an offset to the general expenses of conducting the business; 2nd, as a dividend of 25 per cent. upon participating premiums, on all policies where the *contribution* was *less* than such *percentage;* and, 3rd, the balance was divided amongst the remaining policy-holders, in proportion to their contributions or over-payments.

It is believed that this is the first instance in which a life Company has been enabled to determine what portion of the general surplus arose from the over-payments of participating members, and what other portion resulted from extraneous sources. How accurately this can be done by this new method may be seen by the following practical test.

If there were no extraneous sources of surplus, and no lapses other than by death, it is manifest that the sum total of the several contributions from policy-holders should just equal the total surplus ascertained by a valuation of the policies: thus—

Let P_x = annual premium paid,

p_x = „ cost of insurance (which is generally a *variable*),

$\phi_x = P_x - p_x$ = marginal excess,

c = claims, e = expenses, and Σ the symbol of summation, then we have

$$\Sigma P_x a^n - \Sigma(c+e) = \Sigma A = \text{total assets},$$
$$\Sigma p_x a^n - \Sigma c = \Sigma R'_{x+n} = \Sigma L = \text{total liability},$$
$$\Sigma \phi_x a^n - \Sigma e = \Sigma S = \text{total surplus}.$$

Making $e=0$, we should have, at the end of n years,

$$(\Sigma P_x a^n - \Sigma c) - \Sigma R'_{x+n} = \Sigma S = \Sigma C = \text{sum total of the contributions}.$$

Applying this equation to an assumed practical case, let us suppose a Company to consist of l_{20}, l_{30}, l_{40}, l_{50}, l_{60} members, insured for 1 dollar each, and paying respectively P_{20}, P_{30}, P_{40}, P_{50}, and P_{60} annually—also that the experience during n years is the same as that of the "Mutual Life."* Let us see how the sum total of the contributions, as determined by several methods, will compare with the total surplus. The following results (*see* Tables I., II. and III.) are based upon the experience of the Mutual Life Insurance Company of New York, and 6½ per cent. interest.

It gives me pleasure to say, in conclusion, that whatever credit may attach to the origination and successful application of an entirely new method of dividing surplus, should justly be shared by my young assistant, Mr. David Parks Fackler. Mr. F. has also devised an independent original formula for the same purpose, which, starting from hypotheses substantially the same, produces results nearly identical with the method now described—the one confirming the other. I am in hopes that he will make his very ingenious formula the subject of a paper.

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* I had intended to apply this test to an imaginary Company charging, for instance, premiums by the Carlisle 3 per cent. (properly loaded), and having an experience similar to that shown by the English Life Table, and 4 per cent. interest, but have not been able to do so from want of time. Perhaps one of the younger members of the Institute will feel inclined to undertake this computation as a matter of practice.

TABLE I.—*Amount of Surplus.*

$(1+i)=1{\cdot}065$.	20.	30.	40.	50.	60.
$l_xP_x(1+i)=l_xP_x(1{\cdot}065)$	1,706·42	2,094·97	2,639·29	3,450·71	4,493·07
d_x	723·00	720·00	765·00	962·00	1,546·00
Assets, end of 1st year	983·42	1,374·97	1,874·29	2,488·71	2,947·07
Assets $(1+i)$ . .	1,047·34	1,464·34	1,996·12	2,650·47	3,138·63
$l_{x+1}P_x(1+i)$ . .	1,693·10	2,077·32	2,613·44	3,403·16	4,373·14
Sum	2,740·44	3,541·66	4,609·56	6,053·62	7,511·77
d_{x+1}	722·00	721·00	774·00	1,001·00	1,628·00
Assets, end of 2nd year	2,018·44	2,820·66	3,835·56	5,052·62	5,883·77
Assets $(1+i)$ . .	2,149·64	3,004·00	4,084·57	5,381·04	6,266·22
$l_{x+2}P_x(1+i)$ . .	1,679·80	2,059·64	2,587·29	3,353·67	4,246·84
Sum	3,829·44	5,063·64	6,672·16	8,734·71	10,513·06
d_{x+2}	721·00	723·00	785·00	1,044·00	1,713·00
Assets, end of 3rd year	3,108·44	4,340·64	5,887·16	7,690·71	8,800·06
Assets $(1+i)$ . .	3,310·09	4,622·78	6,269·83	8,190·60	9,372·06
$l_{x+3}P_x(1+i)$ . .	1,666·52	2,041·91	2,560·76	3,302·06	4,113·95
Sum	4,976·61	6,664·69	8,830·59	11,492·66	13,486·01
d_{x+3}	720·00	726·00	797·00	1,091·00	1,800·00
Assets, end of 4th year	4,256·61	5,938·69	8,033·59	10,401·66	11,686·01
Assets $(1+i)$ . .	4,533·29	6,324·50	8,555·78	11,077·77	12,445·60
$l_{x+4}P_x(1+i)$ . .	1,653·25	2,024·11	2,533·83	3,248·12	3,974·31
Sum	6,186·54	8,348·61	11,089·61	14,325·89	16,419·91
d_{x+4}	719·00	729·00	812·00	1,143·00	1,889·00
Assets, end of 5th year	5,467·54	7,619·61	10,277·61	13,182·89	14,530·91
$l_{x+5}R'_{x+5}$ . . .	4,430·11	3,910·47	4,981·42	7,353·43	9,142·71
Surplus . . .	1,037·43	3,709·14	5,296·19	5,829·46	5,388·20

TABLE II.

Age.	ΣS.	Homans. ΣC.	Jellicoe. ΣC. $C=\dfrac{S}{\Sigma\phi a^n}\phi a^n$.	Do. do., Proportioned to Total Surplus.	Percentage on Premiums paid.
20	1,037·43	1,038·53	4,059·06	2,785·66	2,565·97
30	3,709·14	3,710·03	5,277·85	3,622·08	3,146·07
40	5,296·19	5,296·65	6,433·07	4,414·89	3,951·58
50	5,829·46	5,829·93	7,433·17	5,101·24	5,119·63
60	5,388·20	5,388·66	7,776·07	5,336·55	6,477·17
Total	21,260·42	21,263·80	30,979·32	21,260·42	21,260·42

TABLE III.—*Proportions of Surplus.*

	Age 20.			
	Homans.	Jellicoe. $C=\frac{S}{\Sigma\phi a_n}\phi a_n$.	Jellicoe, Proportioned to Total Surplus.	Percentage of Premiums paid.
Deaths during 1st year .	−9·95	7·27	4·99	4·07
„ „ 2nd „ .	−5·12	14·09	9·67	8·13
„ „ 3rd „ .	−·46	20·42	14·01	12·16
„ „ 4th „ .	4·05	26·38	18·10	16·20
„ „ 5th „ .	8·41	31·96	21·94	20·22
Survivors . . .	1,041·60	3,958·94	2,716·95	2,505·19
Total, Age 20 . .	1,038·53	4,059·06	2,785·66	2,565·97
	Age 30.			
Deaths during 1st year .	12·50	10·21	7·00	5·39
„ „ 2nd „ .	17·42	19·83	13·61	10·80
„ „ 3rd „ .	22·31	28·93	19·85	16·25
„ „ 4th „ .	27·18	37·54	25·77	21·74
„ „ 5th „ .	32·05	45·73	31·39	27·29
Survivors . . .	3,598·57	5,135·61	3,524·46	3,064·60
Total, Age 30 . .	3,710·03	5,277·85	3,622·08	3,146·07
	Age 40.			
Deaths during 1st year .	27·89	14·50	9·95	7·89
„ „ 2nd „ .	34·87	28·46	19·53	15·98
„ „ 3rd „ .	41·48	41·97	28·81	24·30
„ „ 4th „ .	48·39	55·15	37·85	32·91
„ „ 5th „ .	55·67	68·11	46·75	41·89
Survivors . . .	5,088·35	6,224·88	4,272·00	3,828·61
Total, Age 40 . .	5,296·65	6,433·07	4,414·89	3,951·58
	Age 50.			
Deaths during 1st year .	39·99	23·78	16·32	14·52
„ „ 2nd „ .	52·07	47·98	32·93	30·23
„ „ 3rd „ .	65·55	72·80	49·97	47·30
„ „ 4th „ .	80·57	98·45	67·56	65·93
„ „ 5th „ .	97·28	125·09	85·85	86·33
Survivors . . .	5,494·47	7,065·07	4,848·61	4,875·32
Total, Age 50 . .	5,829·93	7,433·17	5,101·24	5,119·63
	Age 60.			
Deaths during 1st year .	40·39	49·50	33·97	36·64
„ „ 2nd „ .	70·55	101·07	69·37	77·16
„ „ 3rd „ .	104·71	154·69	106·16	121·78
„ „ 4th „ .	143·00	210·29	144·32	170·64
„ „ 5th „ .	185·50	267·75	183·75	223·89
Survivors . . .	4,844·51	6,992·77	4,798·98	5,847·06
Total, Age 60 . .	5,388·66	7,776·07	5,336·55	6,477·17